A CONVERSE OF THE JORDAN-BROUWER SEPARATION THEOREM IN THREE DIMENSIONS*

BY

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That the closed $(n-1)$-manifold immersed in euclidean space, E_n, of n dimensions $(n>2)$, separates E_n into just two domains of which it is the common boundary, was shown by Brouwer in 1912.† That the points of the manifold are accessible from each of its complementary domains Brouwer proved in an accompanying paper,‡ and in the latter connection he gave an example to show that a bounded and closed point set which separates E_n into just two domains and every point of which is accessible from each of these domains, is not necessarily homeomorphic with a closed manifold.

The above results of Brouwer, in so far as the connectivity of the set residual to the manifold in E_n is concerned, received considerable extension at the hands of J. W. Alexander, who not only demonstrated, using modulo 2 Betti numbers, that the residual set is just two connected domains, but that a certain duality exists between the connectivity numbers of the set and those of its complement.§ If we denote the ith connectivity number of a set F by $R^i(F)$, then, for the particular case where C^i is a set in E_n homeomorphic with an i-sphere, Alexander showed that

$$R^i(C^i) = R^{n-i-1}(E_n - C^i) = 2, \tag{1}$$

$$R^s(C^i) = R^{n-s-1}(E_n - C^i) = 1 \qquad (s\neq i). \tag{2}$$

In the case where $n=3$ and $i=2$, relation (1) states that the complement of a simple closed surface in E_3 is just two domains (as also shown by Brouwer),

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† L. E. J. Brouwer, *Beweis des Jordanschen Satzes für den n-dimensionalen Raum*, Mathematische Annalen, vol. 71 (1912), pp. 314–327.

‡ L. E. J. Brouwer, *Über Jordansche Mannigfaltigkeiten*, ibid., pp. 320–327.

§ J. W. Alexander, *A proof and extension of the Jordan-Brouwer separation theorem*, these Transactions, vol. 23 (1922), pp. 333–349. Familiarity with this work is assumed throughout the present paper.

The terminology "Betti number" is at present being employed more and more to denote connectivity numbers that are uniformly less by one unit than the numbers R^i of the present paper. However, since the present paper is based so much upon the above paper of Alexander, it seems advisable to retain the connectivity numbers as defined in that connection (although they are not termed "Betti numbers" by Alexander).

and relation (2) states the important additional fact that every closed s-chain (where $s<2$) in the complement bounds an $(s+1)$-chain in the complement. Thus, although as Alexander pointed out in another connection* the domains complementary to a simple closed surface in E_3 may not be simply connected (may, in fact, require an infinite number of generators), every closed 1-chain in either of the domains bounds in that domain.

In contemplating so simple a surface as the torus, it becomes apparent that in formulating a *converse* of the Jordan-Brouwer separation theorem the relation (2) must be taken into account as well as relation (1). On the other hand, a glance at Brouwer's example cited above is sufficient to inform one that even with the duality relations (1) and (2), the accessibility of the points of a surface from each of its complementary domains is not enough to ensure that surface being, in E_3, a simple closed surface. Some condition must be imposed that will do away with the crinkliness of the surface. It occurred to the author that such a condition might be the following: If K is a simple closed surface in E_3, D one of the domains complementary to K, and ϵ a positive number, then there exists a positive number δ such that if Γ^0 is a closed 0-chain in D of diameter less than δ, then Γ^0 bounds an open 1-chain Γ^1 in D whose diameter is less than ϵ. This property of D may be concisely expressed by saying that *the* 0-*chains of* D *are uniformly homologous to zero in* D.

It will be shown in §1 of this paper (which may be considered as complementary to Alexander's paper on the Jordan-Brouwer separation theorem, since it utilizes the machinery introduced by him in order to treat accessibility properties into which he did not go), that the above condition is a necessary condition for the general n-dimensional case, and in §3 it will be shown that together with part of Alexander's duality relation it enables one to give, in E_3, a converse of the Jordan-Brouwer theorem.

The chief difficulty encountered by the author in applying the above, essentially combinatorial, properties,† to characterizing the sphere immersed in E_3, was that the known characterizations of the simple closed surface‡

* J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proceedings of the National Academy of Sciences, vol. 10 (1924), pp. 8–10.

† It will be shown below, in the Appendix, that *uniformly homologous to zero* as a condition relative to the 0-chains of a domain is equivalent to the condition *uniform connectedness im kleinen* for the domain.

‡ The earliest characterization of which the author knows was announced by R. L. Moore and J. R. Kline, their abstract being in the Bulletin of the American Mathematical Society, vol. 28 (1922), p. 380. The same definition is given by Miss I. Gawehn (except for the greater generality of the space considered), in *Über unberandete 2-dimensionale Mannigfaltigkeiten*, Mathematische Annalen, vol. 98 (1927–28), pp. 321–354. Cf. also C. Kuratowski, *Une caractérisation topologique de la surface de la sphère*, Fundamenta Mathematicae, vol. 13 (1929), pp. 307–318, as well as abstracts by L. Zippin, Bulletin of the American Mathematical Society, vol. 35 (1929), p. 154 and p. 293.

were all of a topological type which did not seem to lend themselves very readily to association with the combinatorial properties of the complement. Accordingly a new characterization of the simple closed surface by means of internal properties that are readily associated with the combinatorial properties of the complement was worked out, and is given below in §2.

In closing this introduction the author wishes to call to the attention of the reader the problem of giving a converse of the Jordan-Brouwer separation theorem in E_3 by certain accessibility conditions that are more in the spirit of the Schoenflies converse as formulated in E_2,* and it is hoped that if the attempt is made to find such conditions, the present work will be of some assistance as having already broken down some of the barriers. For in the opinion of the author the analysis situs relations between closed sets and their complements in spaces of three and higher dimensions will be most easily discovered by first having regard for the connectivity numbers of the complements,† and although it would be unwise to make any predictions, it would certainly seem as though to disregard these in the case of the $(n-1)$-sphere, when already known, in the attempt to obtain equivalent accessibility conditions, would involve an extreme waste of energy.

1. We shall show in this section‡ that if M^{n-1} is an $(n-1)$-sphere im-

* In this connection see J. R. Kline, *Separation theorems and their relation to recent developments in analysis situs*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 155–192. See especially pp. 156–159 and top of p. 191. As will be observed later on, *regular* accessibility is an immediate consequence of the conditions given for the converse in the present paper.

† The recent work of P. Alexandroff bears out this opinion strikingly. Cf. his *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Annals of Mathematics, (2), vol. 30 (1928), pp. 101–187. The author wishes to seize this opportunity to express his indebtedness to Dr. Alexandroff, contact with whom, both through his memoirs and personally (during a recent brief visit at the University of Michigan), directed the attention of the author to the possibility of associating continuity and combinatorial methods in treating the problems of analysis situs.

‡ The content of this section was presented, along with certain other results, to this Society, March 29, 1929, under separate title (cf. Bulletin of the American Mathematical Society, vol. 35, p. 458, abstract No. 22). It is felt, however, that the theorems and corollaries given above properly belong in the present paper. As for the other results of the paper just referred to, the theorem that the points of an i-cell $(0 \leqq i \leqq n-1)$ are regularly accessible from the complement in E_n is a special case of a more recent theorem established by the author, to the effect that in E_n the points of any closed set which is homeomorphic with a subset of E_{n-1} are regularly accessible from the complement; several of the other results were found to duplicate already known theorems and were withdrawn from the announcement in abstract. One of the things so deleted was an example which the author believes to be not without interest, and which he has not seen in the literature, viz., an arc in E_3 which cannot lie on, or bound, any set homeomorphic with a 2-cell, and which cannot be thrown into a straight line interval by a (1-1)-continuous transformation of E_3 into itself. (Cf. L. Antoine, *Sur l'homéomorphie de deux figures et de leurs voisinages*, Journal de Mathématiques, (8), vol. 4 (1921), pp. 221–325.) The chief interest of the example is its ease of description: On a great circle of the 2-sphere, let $ABCDEA$ be points in the cyclical order named, and let the portion BC of the

mersed in E_n,* and D is either of its complementary domains, then the 0-chains of D are uniformly homologous to zero in D. Consequently the domain D is uniformly connected im kleinen (see Appendix)—a fact which was established for the case where $n=2$ and M^{n-1} is a 1-sphere by R. L. Moore.† As a corollary it follows that the points of the sphere are not only accessible, but are regularly accessible, from the complementary domains.

THEOREM 1. *Let M^{n-1} be an $(n-1)$-sphere immersed in E_n, and let D be one of the domains of $E_n - M^{n-1}$. Then the 0-chains of D are uniformly homologous to zero in D.*

Let the domain $E_n-(M^{n-1}+D)$ be denoted by D_1.

Suppose the 0-chains in D are not uniformly homologous to zero. Then it easily follows that there is a point P on M^{n-1} and a spherical neighborhood U of P, such that in every neighborhood of P there is a 0-chain which does not bound in that part of D which lies in U. We shall denote the frontier of U by F.

There is a cell C^{n-1} of M^{n-1} which contains P as a non-boundary point and lies wholly in U. Denote the boundary of C^{n-1} by C^{n-2}, and let the set of all those points of M^{n-1} that are not interior points of C^{n-1} be denoted by B^{n-1}. Let V be a spherical neighborhood of P of such a radius that V contains no points of B^{n-1}. Let x_1^0 and x_2^0 be any two 0-cells of D that lie in V. Since the closed chain $x_1^0+x_2^0$ bounds a 1-chain $L_B{}^1$ interior to V, we have

$$x_1^0 + x_2^0 \sim 0 \qquad [\text{mod } 2, E_n - (F + B^{n-1})]. \tag{3}$$

In order to deal with a specific case, we may assume that a 0-cell, y_1^0, of $L_B{}^1$, lies in D_1, $L_B{}^1$ having been obtained, say, by joining x_1^0 and x_2^0 to such a cell by chains $L_{B_1}^1$ and $L_{B_2}^1$, respectively.

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arc $ABCD$ be replaced by a new portion containing two infinite sequences of simple trefoil knots (i.e., knots in the sense that the great circle becomes knotted), which of course have to leave the 2-sphere, and which respectively approach the points B and C as limiting points. This can be done in such a way that the new portion BC is still an arc from B to C that does not meet the arc $BAED$ of the great circle. Then the portions AB and CD of the great circle, together with the new portion BC, is an arc of the type described above.

* I.e., M^{n-1} is a point set which is in (1-1)-continuous correspondence with an $(n-1)$-sphere, and is therefore not necessarily formed by cells of the subdivisions of E_n which are used in the congruences and homologies of the proof. Cf. J. W. Alexander, these Transactions, loc. cit., for definitions of *chain*, *homology*, etc. The reader should be on his guard in the matter of terminology which has various meanings depending upon its application in a point-set theoretic sense or in a combinatorial sense. (E.g., "chain," "closed," "open," etc. Present usage seems to apply the term "chain" only to open chains, chains that are closed being termed "cycles." We are retaining the terminology of Alexander's paper, however, in the present connection.)

† R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–370.

On the other hand, as every 0-chain in D bounds in D, even though it may not bound im kleinen, there is a chain L_C^1 in D which is bounded by the chain $x_1^0+x_2^0$. Hence,

$$x_1^0 + x_2^0 \sim 0 \qquad (\text{mod } 2,\ E_n - C^{n-1}). \tag{4}$$

If now we can show that the closed chain

$$L_B^1 + L_C^1 \sim 0 \qquad (\text{mod } 2,\ E_n - C^{n-2}), \tag{5}$$

it will follow from relations (3), (4) and (5), and Alexander's Corollary W^i,* that $x_1^0+x_2^0$ bounds in the set common to D and U, and since no restriction is placed upon the choice of x_1^0 and x_2^0, a contradiction of the supposition that the 0-chains of D are not uniformly homologous to zero will be obtained. We therefore proceed to demonstrate the validity of relation (5).

Let C_B^{n-1} be an arbitrary cell of M^{n-1} interior to B^{n-1}. Then x_1^0 and y_1^0 bound a 1-chain K_1^1 in $E_n-(M^{n-1}-C_B^{n-1})$. The closed chain $K_1^1+L_{B1}^1$ links C^{n-2}, for if it did not, then, by virtue of Alexander's Corollary W^i, $x_1^0+y_1^0$ would bound in E_n-M^{n-1}. Similarly, the closed chain $K_1^1+L_C^1+L_{B2}^1$ links C^{n-2}. However, by relation (1) there is only one linearly independent non-bounding 1-chain in E_n-C^{n-2}. Hence

$$K_1^1 + L_{B1}^1 \sim K_1^1 + L_C^1 + L_{B2}^1 \qquad (\text{mod } 2,\ E - C^{n-2}). \tag{6}$$

Since homology (6) implies homology (5), the theorem is proved.

COROLLARY 1. *The points of an $(n-1)$-sphere, M^{n-1}, immersed in E_n, are regularly accessible*† *from the complement of the sphere.*

Let P be a point of M^{n-1}, and D one of the domains complementary to M^{n-1}. Since there are only two domains in the complement of M^{n-1}, it will be sufficient to prove P regularly accessible from D.

By virtue of Theorem 1 there exists (1) a sequence of spherical neighborhoods of P, $U_1, U_2, U_3, \cdots$, such that if r_k is the radius of U_k, then $r_k > r_{k+1}$, and $\lim_{k\to\infty} r_k = 0$; and such that for $k>1$, any closed 0-chain of D that lies in U_k bounds a 1-chain that lies wholly in the common part of U_{k-1} and D; (2) a sequence of 0-cells $x_2^0, x_3^0, x_4^0, \cdots$, such that, for every k, x_k^0 lies in the common part of D and U_k.

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* These Transactions, loc. cit., p. 342.

† A point P is said to be *regularly accessible* from a point set D if for every positive number ϵ there exists a positive number δ such that if Q is a point of D whose distance from P is less than δ, then there is an arc PQ in $D+P$ whose diameter is less than ϵ. Cf. G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 650–657.

For any k, x_k^0 and x_{k+1}^0 bound a 1-chain, Γ_k^1, which lies wholly in the common part of D and U_{k-1}. As the set of points

$$M = \sum_{k=2}^{\infty} \Gamma_k^1 + P$$

is a continuous curve, there is an arc from x_2^0 to P that lies in M, and hence (except for P) in the common part of U_1 and D. As the first neighborhood, U_1, is arbitrary, and as in every neighborhood of a point x of D in U_2 there can be found a 0-cell x_2^0 in D, the corollary is proved.

If, in the proof of Theorem 1, M^{n-1} denotes an $(n-1)$-manifold, and P denotes a point which is interior to (i.e., not on the boundary of) a cell C^{n-1} which is one of the cells defining M^{n-1}, then the argument to show that $x_1^0 + x_2^0$ is homologous to zero in $D \cdot U$ goes through as before. However, if P is such a point of the manifold that it cannot lie interior to any such cell, then the above 0-chain may fail to bound in $D \cdot U$. (Cf., for instance, the set K defined in the Appendix, paragraph beginning "Condition (3)".) Hence we can state the following theorem:

THEOREM 1a. *Let M^{n-1} be any $(n-1)$-manifold immersed in E_n, D a domain complementary to M^{n-1}, and P a point of M^{n-1} which can be considered as interior to an $(n-1)$-cell of definition of M^{n-1}. Then, if C^{n-1} is any such cell, and U is any spherical neighborhood of P that encloses only points of M^{n-1} that are interior to C^{n-1}, there exists a neighborhood V of P such that any 0-chain in $D \cdot V$ bounds a 1-chain in $D \cdot U$. If every point of M^{n-1} can be considered as lying interior to an $(n-1)$-cell of definition of M^{n-1}, the 0-chains of D are uniformly homologous to zero in D.**

COROLLARY 2. *The points of an $(n-1)$-manifold, M^{n-1}, immersed in E_n, are regularly accessible from the complement of the manifold.*

If P is an interior point of a cell C^{n-1} of M^{n-1}, the proof is the same as that of the above Corollary 1. If not, then P is on the boundaries of a certain number (at least four) of $(n-1)$-cells of M^{n-1}. Let V be any neighborhood of P. Since any $(n-1)$-cell of an $(n-1)$-manifold is homeomorphic with an $(n-2)$-sphere and its interior in E_{n-1}, there is a neighborhood U of P such that any point of M^{n-1} which lies in U and is interior to a cell C^{n-1} of M^{n-1}

* In order to avoid confusion in the mind of the reader, it should perhaps be pointed out here that we are using the definition of manifold as given by Alexander in his paper on the Jordan-Brouwer Separation Theorem. That is, an i-manifold is a closed i-chain whose ith connectivity number has the value 2. Some authors (cf., for instance, Veblen's *Analysis Situs*, published by this Society, 1922, p. 88) state the additional condition that every point of the i-manifold shall have a neighborhood, relative to the manifold, which is an i-cell. The definition as given by Alexander of course yields greater generality to the results obtained.

is joined to P by an arc of $M^{n-1}\cdot V$ that lies, except for P, wholly interior to C^{n-1} (by Corollary 1). Let Q be any point of D (one of the domains of E_n-M^{n-1}) that lies in U. We shall show that there is an arc from P to Q lying, except for P, wholly in $D\cdot V$.

Let PQ be any arc of U from P to Q, and let P_1 be its first point on M^{n-1}, in the order from Q to P. Since in any neighborhood of a point of M^{n-1} there are points of M^{n-1} not on the boundaries of its $(n-1)$-cells, we may assume that P_1 is interior to a cell C^{n-1} of M^{n-1} whose boundary, C^{n-2}, contains P. There is an arc, t, of M^{n-1}, from P_1 to P and lying wholly interior to C^{n-1}, except for P, as well as interior to V. Let $P_2, P_3, P_4, \cdots$ be a sequence of points of t having P as a sequential limit point, and let ϵ denote the distance between t and the frontier of V as ordinarily defined for point sets.

Consider the portion P_1P_2 of t. Since all points of P_1P_2 are interior points of C^{n-1}, it is easy to show that there lies, in $D\cdot V$, an arc Q_1Q_2 approximating P_1P_2, where Q_1 is a point of QP_1 in the neighborhood of P_1, and Q_2 a point in the neighborhood of P_2. For if x is any point of P_1P_2, there is a number δ such that any 0-chain of D in $S(x, \delta)$ bounds a 1-chain of D in $S(x, \frac{1}{2}\epsilon)$ (Theorem 1a). A simple chain* of the neighborhoods $S(x, \delta)$ from P_1 to P_2, together with the joining of 0-cells in the successive links of the simple chain, yields the arc Q_1Q_2 desired.

By successive approximations to the arcs P_iP_{i+1} by arcs Q_iQ_{i+1} in $D\cdot V$ in such a manner that if x_i is any point of Q_iQ_{i+1}, $\lim_{i\to\infty} x_i=P$, it is clear that the set $QQ_1+\sum_{i=1}^{\infty} Q_iQ_{i+1}+P$ is a continuous curve containing an arc from Q to P and lying, except for P, wholly in $D\cdot V$.

2.† We shall now prove the following theorem:

THEOREM 2. *In a metric separable space let M be a compact set containing at least one simple closed curve, and satisfying the following conditions*: (1) *if t is an arc of M, then $M-t$ is connected*; (2) *if J is a simple closed curve of M, then $M-J$ is the sum of two uniformly connected im kleinen‡ components. Then M is a simple closed surface*; *i.e., a set homeomorphic with the sphere* $x^2+y^2+z^2=1$.§

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* Cf. R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 131–164. In particular, see bottom of p. 134, and Theorem 10 of this paper.

† The content of this section was presented to the Society December 31, 1928.

‡ A set of points K is said to be *uniformly connected im kleinen* if for every $\epsilon>0$ there exists a $\delta>0$, such that if P and Q are points of K whose distance apart is less than δ, then P and Q are in a connected subset of K whose diameter is less than ϵ. However, for an open subset of a continuous curve, it is easy to prove that the words "a connected subset" in this definition may be replaced by the words "an arc."

§ When this paper was in process of completion, Dr. Leo Zippin communicated to the author that he had succeeded in showing that a continuous curve C which has the properties that (1) if

For ease of reference, we shall divide the proof into sections:

I. *The set M is connected im kleinen.* For if J is a simple closed curve of M, and M_1 and M_2 are the components of $M-J$, then M_1+J and M_2+J are connected im kleinen by a theorem of R. L. Moore.* Accordingly the set M is itself connected im kleinen.

II. *If J is a simple closed curve of M, and M_1 and M_2 are the components of $M-J$, then J is identical with the boundary of each of the sets M_1, M_2.* For suppose J contains a point, P, which is not a limit point of M_1, say. Then an arc, t, of J, contains all the boundary points (if any) of M_1. Then

$$M - t = M_1 + (M_2 + J - t).$$

Since M_1 and M_2+J-t are mutually separated sets, a violation of condition (1) results.

III. *The set M is connected.* For M contains at least one simple closed curve, J, and, by II, J is the common boundary of the components of $M-J$.

IV. *If P is a point of M, then there is an arc of M which contains P, and of which P is not an end point.*†

Since M is a continuous curve, being a connected im kleinen, closed and connected set, there is an arc PP' of M, joining P to some point P' of $M-P$. By condition (1) the set $M-PP'=D$ is connected. If P is not a limit point of D, then there is a point Q of PP' distinct from both P and P', such that the portion PQ of PP' contains no limit points of D. Then the set

$$M - P'Q = D + (PQ - Q)$$

is not connected, since D and $PQ-Q$ are mutually separated sets. But this constitutes a violation of condition (1). Consequently D has P as a limit point.

The set $D+P$ is connected im kleinen. If not, then P is the only point at which it is not connected im kleinen, since D is an open subset of M. Then there exists, by Theorem 1 of my paper *Characterizations of continuous curves that are perfectly continuous*,‡ a positive number ϵ such that for every positive

t is an arc of C, then $C-t$ is connected, (2) if J is a simple closed curve of C, then $C-J$ is not connected, is a simple closed surface. This definition is an obvious improvement over that given above, and by its use the last two parts of the proof of the theorem of §3 can be omitted.

* R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fundamenta Mathematicae, vol. 3 (1922), pp. 232–237, Theorem 1. The proof given by Moore evidently holds for a compact set in any topological space.

† Note added in proof: Since this paper was submitted to these Transactions, a paper by W. L. Ayres has appeared in the American Journal of Mathematics for October, 1929, in which it is shown (Theorem 10) that if a continuous curve in E_n contains no cut point, then the curve is cyclicly connected. As our curve M clearly contains no cut point by virtue of condition (1), it is obvious that the results IV and V follow from Ayres' theorem.

‡ Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 614–621.

number $\delta < \epsilon$ there is a point Q of D whose distance from P is less than δ and which is not in the same quasi-component with P of the set of points $(D+P) \cdot S(P, \epsilon)$.* It follows at once that there is a sequence of points $x_1, x_2, x_3, \cdots$, such that $\epsilon/2 > \rho(x_n, P) > \epsilon/3$, and such that no two points of this sequence lie in the same quasi-component of $(D+P) \cdot S(P, \epsilon)$. As M is compact, there is a limit point, x, of the set of points $\sum_{n=1}^{\infty} x_n$. The point x cannot lie in D, since D is connected im kleinen. Hence x is on the arc PP', and since there is no loss of generality in assuming that $\rho(P, P') > \epsilon$, we can say that x is distinct from both P and P'.

However, D is uniformly connected im kleinen in the neighborhood of x. For let t be an arc in PP' whose end points are distinct from P and P', and which contains x. Since $M-t$ is connected, by condition (1), there exists in $M-t$, by a theorem of R. L. Moore,† an arc from P to P', and this arc together with the arc PP' contains a simple closed curve J which contains t, consisting of two arcs, AxB of PP' and AyB of $D+A+B$. Let the components of $M-J$ be M_1 and M_2, and let η be a positive number such that $S(x, \eta)$ contains no points of $J+PP'$ that are not points of t. Then by condition (2) there is a positive number ρ such that every two points of M_1 that lie in $S(x, \rho)$ are joined by an arc of M_1 in $S(x, \eta)$, and a similar statement holds for M_2. Since all points of $M-J$ in $S(P, \eta)$ belong to D, it is clear that if η is taken so small that $S(x, \eta)$ is a subset of $S(P, \epsilon)$, the points $x_1, x_2, x_3, \cdots$ cannot all lie in distinct quasi-components of $(D+P) \cdot S(P, \epsilon)$.

Accordingly $D+P$ must be connected im kleinen, and by a theorem of Whyburn‡ P is regularly accessible from D. Hence there is an arc of M containing P and of which P is not an end point.

V. *If P is a point of M, then P is on some simple closed curve of M.* This follows at once from IV, and the method used in IV to obtain the simple closed curve J.

VI. *If P is a point of M, $M-P$ is connected.* By V, P is on a simple closed curve J of M, and by condition (2) $M-J$ is the sum of two components which have, by II, J as common boundary. Since $M-P$ is the sum of these

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* Hereafter, if P is a point and ϵ any positive number, the symbol $S(P, \epsilon)$ will denote a spherical neighborhood of P of radius ϵ; i.e., the set of all points of the space whose distance from P is less than ϵ. In accordance with the usual notation, $M \cdot N$ denotes the set of points common to M and N. By $\rho(P, Q)$ we denote the distance between two points P and Q.

† R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 1. The theorem is evidently true for the case of a continuous curve in the general space we are considering.

‡ G. T. Whyburn, *Concerning accessibility in the plane and regular accessibility in n dimensions*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 504–510, Theorem 3. Although stated for E_n, the proof given for this theorem clearly holds in any metric space.

components together with some of their common boundary points, it is clear that $M-P$ is connected.

VII. *If* M_1, M_2 *and* J *are defined as in* II, *and* AxB *is an arc such that* $<AxB>$* *is in* M_1 *and* $A+B$ *on* J, *then* $M_1 - <AxB>$ *is the sum of two components* R_1 *and* R_2 *whose boundaries are, respectively,* $AxB+$*an arc* AyB *of* J, *and* $AxB+$*an arc* AzB *of* J, *where* AyB *and* AzB *have only* A *and* B *in common.*

Let the two arcs into which A and B divide J be AyB and AzB. Let J_1 and J_2 be simple closed curves defined as follows:

$$J_1 = AxB + AyB,$$

$$J_2 = AxB + AzB.$$

By condition (2), $M-J_1$ is the sum of two components R_1 and R_1' whose common boundary, by II, is J_1. Let R_1 be that one of these components which lies in M_1. (It is clear that M_2 lies in one of the components of $M-J_1$, say in R_1', and that R_1' must therefore contain $<AzB>$, thus requiring that R_1 be a subset of M_1.) Let that one of the components of $M-J_2$ that lies in M_1 be denoted by R_2.

Suppose that M_1 contains a point P that is not in the set $R_1+R_2+<AxB>$. Then P lies in R_1', and if P_1 is a point of M_2, there is an arc PP_1, from P to P_1, in R_1'. If Q is the first point of J on this arc, in the order from P to P_1, then $PQ-Q$ lies in M_1. It is clear that Q is in $<AzB>$; let $S(Q, \epsilon)$ be a neighborhood of Q that does not enclose any point of AxB. There is a neighborhood $S(Q, \delta)$ such that any two points of $M_1 \cdot S(Q, \delta)$ are joined in $S(Q, \epsilon)$ by an arc of M_1. Since points of $PQ-Q$ and R_2 both lie in $S(Q, \delta)$, and such points are then joined by connected subsets of $M-J_2$, a contradiction results from the supposition that M_1 contains points not in the set $R_1+R_2+<AxB>$. Hence $M_1 = R_1+R_2+<AxB>$.

VIII. *If* P *is a point of* M, *and* ϵ *is any positive number, there is a simple closed curve* J *in* M *which does not contain* P *and such that both* J *and the component of* $M-J$ *which contains* P *lie in* $S(P, \epsilon)$.

By V, P is on some simple closed curve J of M. Denote the components of $M-J$ by M_1 and M_2. We can, of course, suppose that J is not wholly in $S(P, \epsilon)$, and that both M_1 and M_2 have points in the exterior of $S(P, \epsilon)$.

By virtue of the uniform connectedness im kleinen of M_1 and M_2, together with II, it is easy to show the existence, in $S(P, \epsilon)$, of arcs A_1xB_1 and A_2yB_2, where $<A_1xB_1>$ and $<A_2yB_2>$ lie in M_1 and M_2, respectively, and the points A_1, A_2, B_1, B_2 lie on J in the order $A_1A_2PB_2B_1A_1$, and such that the arc A_1PB_1 of J lies in $S(P, \epsilon)$. Let the simple closed curve $A_1xB_1+B_1B_2+A_2yB_2$

* By $<AxB>$ we denote the set $AxB-A-B$.

$+A_1A_2$ (where A_1A_2 and B_1B_2 are portions of A_1PB_1 on J) be denoted by J_1. Let that component of $M-J_1$ that contains P be denoted by R_1.

By VII, $M_1-<A_1xB_1>$ is the sum of two components, one of which is bounded by the simple closed curve $T_1=A_1PB_1+A_1xB_1$; denote this component by K_1. Similarly, that component of M_2 bounded by the simple closed curve $T_2=A_2PB_2+A_2yB_2$ denote by K_2. Then

$$R_1 = K_1 + K_2 + < A_2PB_2 >.$$

For by VII, $R_1-<A_2PB_2>$ is the sum of two components H_1 and H_2, bounded, respectively, by T_1 and T_2. Now T_1 bounds only two components, one of which contains M_2. If H_1 were the latter component, then H_1 would contain $<A_2yB_2>$, which is clearly impossible. Then H_1 is that component of $M-T_1$ that does not contain M_2, i.e., K_1. Similarly, $H_2=K_2$. We note that if α is a point of M_1 that is not in K_1+T_1, then R_1 does not contain α.

If R_1 does not lie wholly in $S(P, \epsilon)$, there is a point, P_1, in R_1, such that $\rho(P, P_1)=\epsilon$.

In a similar manner, we can show the existence, in $S(P, \epsilon/2)$, of arcs $C_1x_1D_1$ and $C_2y_2D_2$, where $<C_1x_1D_1>$ and $<C_2y_2D_2>$ lie in K_1 and K_2, respectively, and the end points of these arcs lie on A_2PB_2 in the order $A_2C_1C_2PD_2D_1B_2$. Denote the simple closed curve formed by these two arcs, together with the portions C_1C_2 and D_1D_2 of A_2PB_2, by J_2. Denote the component of $M-J_2$ that contains P by R_2.

By VII, $K_1-<C_1x_1D_1>$ is the sum of two components, one of which, N_1, is bounded by the simple closed curve $F_1=C_1x_1D_1+C_1PD_1$; similarly, that component of $K_2-<C_2y_2D_2>$ bounded by the curve $F_2=C_2y_2D_2+C_2PD_2$ denote by N_2. Then we can show, as above, that

$$R_2 = N_1 + N_2 + < C_2PD_2 >.$$

Consequently, R_2 is a subset of R_1, and does not contain the point α defined above.

We may continue in this way indefinitely, obtaining simple closed curves J_3, J_4, J_5, $\cdots$, with corresponding components R_3, R_4, R_5, $\cdots$, all containing P, but not α, and each component a subset of the preceding; also, requiring that for each positive integer n, the curve J_n lie in $S(P, \epsilon/n)$.

For some n, R_n is a subset of $S(P, \epsilon)$. For if not, there is for every n a point P_n of R_n such that $\rho(P, P_n)=\epsilon$. Since M is compact, the set of points $\sum_{n=1}^{\infty} P_n$ has at least one limit point Q, which is in every set R_n and such that $\rho(P, Q)=\epsilon$. Common to all the sets R_1, R_2, R_3, $\cdots$, there is a continuum C, which contains P and Q.

By VI, $M-P$ is connected, and therefore contains an arc t with end points Q and α.* There exists a positive integer k such that ϵ/k is less than the distance between P and the arc t. Then the set $C+t$ is a connected subset of $M-J_k$, since C lies in R_k and t cannot meet J_k. However, P lies in R_k and α lies in $M-(J_k+R_k)$. Thus the supposition that for no n does R_n lie in $S(P, \epsilon)$ leads to a contradiction.

The conclusion of Theorem 2 now follows in either one of the following ways: (1) by virtue of the Moore-Kline-Gawehn† definition of the simple closed surface, whose conditions are now seen to be fulfilled; or (2) by virtue of R. L. Moore's Axioms Σ_1 for plane analysis situs.‡ For it is easy to see, in view of what has been shown above, that if Q is an arbitrary point of M, then $M-Q$ is a topological plane, in that it satisfies the axioms Σ_1. Thus, if J is a simple closed curve of $M-Q$, let that component of $M-J$ which does not contain Q be called a region.

3.§ We shall prove the following theorem:

THEOREM 3 (Converse of the Jordan-Brouwer Separation Theorem in E_3). *Let K be a closed and bounded set in E_3, such that $E_3-K=S_1+S_2$, where S_1 and S_2 are mutually exclusive and*

(1) *every arc from a point of S_1 to a point of S_2 contains at least one point of K;*

(2) *if P is a point of K and Q a point not in K, then in every neighborhood of P there is a point P' such that there is an arc from P' to Q lying except for possibly P' wholly in E_3-K;*

(3) *the 0-chains of $S_i (i=1, 2)$ are uniformly homologous to zero in S_i; and*

(4) *the Betti number* (mod 2) $R^1(E_3-K)=1$.

Then K is a simple closed surface.

We shall show that the set K satisfies all the conditions of Theorem 2.

I. *The set K is connected.* For if not, it is the sum of two mutually separated sets, K_1 and K_2. Let P_i be a point of K_i $(i=1, 2)$. By a theorem of Knaster and Kuratowski,¶ there exists a continuum C in E_3-K which separates P_1 and P_2. The continuum C lies wholly in one of the sets S_1, S_2.

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* Cf. R. L. Moore, Mathematische Zeitschrift, loc. cit.

† Cf. Bulletin abstract of Moore and Kline, loc. cit., and I. Gawehn, loc. cit.

‡ R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 131–164. See also R. L. Wilder, *Concerning R. L. Moore's axioms Σ_1 for plane analysis situs*, Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 752–760.

§ The contents of this section were presented to the Society August 30, 1929.

¶ B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), pp. 206–255. The theorem referred to here is Theorem 37, p. 233. This theorem holds true in E_n, since the theorem of Brouwer used in its proof holds in E_n, for both bounded and unbounded sets.

For if it contains points of both S_1 and S_2, then S_1, say, contains a limit point of S_2. But then, since K is closed, this limit point would be joined to a point of S_2 by an arc that does not meet K, a violation of condition (1). Then C lies in S_1, say.

Now in the neighborhood of any point P of K there lie points of both S_1 and S_2. For if we let U be any neighborhood of P and x a point of S_1, say, there is, by condition (2), a point P' in U which is joined to x by an arc $P'x$ and such that $P'x-P'$ lies in E_3-K. It is clear that $P'x-P'$ must lie in S_1, since x is in S_1; and as U must contain points of $P'x-P'$, there is a point of S_1 in U.

Accordingly, in the same domain complementary to C that contains P_1 there is a point x_1 of S_2. By condition (2) there is, in the same domain complementary to C as P_2, a point x_2 such that there is an arc x_1x_2 which lies, except possibly for x_2, wholly in E_3-K. Then $x_1x_2-x_2$ lies in S_2, and there is no point of C on x_1x_2, since C is in S_1. But x_1 and x_2 must be separated by C in E_3. Thus the supposition that K is not connected leads to a contradiction.

II. *The set K is connected im kleinen.* For if not, it follows from a theorem of R. L. Moore* that there exist two concentric spheres, R_1 and R_2, and a sequence of subcontinua of K, namely M_∞, M_1, M_2, $M_3, \cdots$, such that (1) each of these continua contains at least one point of R_1 and R_2, respectively, but no point exterior to R_1 or interior to R_2, (2) no two of these subcontinua have a point in common, and no two of them contain points of any connected subset of K that lies wholly in R_1+R_2+I, where I is the annular domain bounded by R_1 and R_2, (3) M_∞ is the sequential limiting set of the sequence of continua $M_1, M_2, M_3, \cdots$.

Let P be a point of M_∞ in I, and let R be a spherical neighborhood of P such that R' lies in I (where R' denotes R together with its boundary). Let us consider R as a space, $\overline{E}_3$, and if M is a point set, let us denote the product $M\cdot R$ by $\overline{M}$.

There exists a positive integer m such that $\overline{M}_i$, for $i\geqq m$, is non-vacuous. The set $\overline{K}$ is closed relative to $\overline{E}_3$, and by condition (2) of Moore's theorem just quoted, no connected subset of $\overline{K}$ joins $\overline{M}_i$ and $\overline{M}_j$ ($i\neq j$; $i, j\geqq m$). There exists a spherical neighborhood, U, of P in $\overline{E}_3$, such that any 0-chain of S_i in U is homologous to zero in $\overline{S}_i$. Let i and j be values such that $\overline{M}_i$

* R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy of Sciences, vol. 4 (1918), pp. 364–370. The theorem referred to here is not given any explicit statement, in theorem form, in this paper, but will be found in such form in the same author's *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 296–297.

and $\overline{M}_j$ have points P_i and P_j, respectively, in U. There exists a continuum, C (continuum relative to $\overline{E}_3$) that lies in $\overline{E}_3-\overline{K}$ and separates P_i and P_j in $\overline{E}_3$. For there is a separation of $\overline{K}$ into two mutually separated sets containing $\overline{M}_i$ and $\overline{M}_j$, respectively,* and the theorem of Knaster and Kuratowski used in I applies in $\overline{E}_3$.

Now, as shown in a similar case in I, C must lie wholly in, say, S_1. But in every neighborhood of P_i, and of P_j, there are points of S_2; in particular, there are such points in neighborhoods of P_i and P_j that lie in $U-C\cdot U$, and these bound 1-chains of $\overline{S}_2$. As such chains must contain points of C, and hence of S_1, a contradiction is established, and K must be connected im kleinen.

III. *The sets S_1 and S_2 are connected.* Consider S_1, and let x and y be any two of its points. Let P be any point of K, and let U_1 be a spherical neighborhood of P. By condition (3) there is a spherical neighborhood U_2 of P such that if a^0 and b^0 are two 0-cells of S_1 in U_2, then a^0 and b^0 bound a 1-chain of S_1 in U_1. Now, as already shown, U_2 contains two points, x_1 and y_1, which are joined to x and y, respectively, by arcs xx_1 and yy_1 in S_1. There is no loss of generality in assuming that x_1 and y_1 are 0-cells of the subdivision of E_3, and accordingly bound a 1-chain L^1 of S_1 in U_1. Then $xx_1+L^1+yy_1$ is a connected subset of S_1 joining x and y, and therefore S_1 is connected.

IV. *No arc of K disconnects K.* For let t be an arc of K and suppose that $K-t=K_1+K_2$, where K_1 and K_2 are mutually separated sets. The sets K_1+t and K_2+t are closed.

Let P and Q be points of S_1 and S_2, respectively, which are also 0-cells of the subdivision of E_3. By condition (2), K_1+t does not separate P and Q in E_3; let L_1^1 be a 1-chain bounded by P and Q in the complement of K_1+t. Similarly, let L_2^1 be a 1-chain bounded by P and Q in the complement of K_2+t. Since the closed 1-chain $L_1^1+L_2^1$ cannot link the arc t,† it follows from a theorem of J. W. Alexander† that $P+Q$ bounds a 1-chain in E_3-K; in other words, condition (1) is violated. Hence t does not separate K.

V. *If J is a simple closed curve of K, then $K-J$ is not connected.* Suppose that $K-J$ is connected.

By the Alexander duality relation (see introduction) there is a closed 1-chain Γ^1 which links J. We shall first show that Γ^1 has points in both S_1 and S_2.

Suppose Γ^1 is contained in S_1+K. Then it is clear that Γ^1 contains points of K, for if not, by condition (4) we should have

* This follows from the fact that the closed set $K\cdot R'$ is separated between $M_i\cdot R'$ and $M_j\cdot R'$.

† J. W. Alexander, these Transactions, loc. cit.

$$\Gamma^1 \sim 0 \qquad (\text{mod } 2,\ E_3 - J),$$

which is impossible since Γ^1 links J. Let the distance between J and Γ^1, as ordinarily defined for point sets, be denoted by ϵ. By condition (3) of the theorem, there is a number δ_ϵ such that if Γ^0 is a closed 0-chain of S_1 of diameter less than δ_ϵ, then there is a 1-chain Γ_1^1 such that

$$\Gamma_1^1 \equiv \Gamma^0 \qquad (\text{mod } 2,\ S_1)$$

where the diameter of Γ_1^1 is less than $\frac{1}{4}\epsilon$.

We may assume, without loss of generality, that the 1-chain Γ^1 is irreducible; and that its 1-cells have a given cyclic order. We may also assume that the one-cells of Γ^1 are all of diameter less than $\frac{1}{4}\delta_\epsilon$.

The cells of Γ^1 may be divided into two classes, according to whether their boundaries lie wholly in S_1 or not. We may start with a given cell c^1, and consider the successive cells of Γ^1 in their cyclic order. If both end-cells of c^1 lie in S_1, then c^1 may be replaced by a 1-chain K^1 which has the same boundary, but which lies entirely in S_1 and has a diameter less than $\frac{1}{4}\epsilon$. We may agree to say that the cell c^1 has been transformed into the *chain-cell* K^1. Proceeding to the next 1-cell of Γ^1, which we may denote by d^1, suppose that one of its end-cells is on K. Denote the cell that it has in common with c^1 by a^0, and the cell on K by b^0. There is a 0-cell, $\bar{b}^0$, in S_1, whose distance from b^0 is less than $\frac{1}{4}\delta_\epsilon$, and since the distance from a^0 to $\bar{b}^0$ is less than δ_ϵ, these two cells bound a 1-chain K_1^1 in S_1, whose diameter is less than $\frac{1}{4}\epsilon$. We shall replace b^0 by $\bar{b}^0$, and d^1 by K_1^1, and say that b^0 and d^1 have been transformed, respectively, into $\bar{b}^0$ and K_1^1. Suppose the next cell on Γ^1 is e^1, and that its end-cell distinct from b^0, viz. c^0, is also on K. There is a 0-cell $\bar{c}^0$ in S_1 whose distance from c^0 is less than $\frac{1}{4}\delta_\epsilon$. Clearly the distance from $\bar{b}^0$ to $\bar{c}^0$ is less than δ_ϵ, and hence these two cells bound a 1-chain K_2^1 in S_1 of diameter less than $\frac{1}{4}\epsilon$. We shall let $\bar{c}^0$ and K_2^1 be the transforms, respectively, of c^0 and e^1.

Proceeding through all of the cells of Γ^1 as just indicated, we replace each cell of Γ^1 by its transform, and obtain a new closed 1-chain, Γ_1^1, which we may call the transform of Γ^1. Now Γ_1^1 lies wholly in S_1, and accordingly, by application of condition (4),

$$\text{(a)} \qquad \Gamma_1^1 \sim 0 \qquad (\text{mod } 2,\ E_3 - J).$$

We shall show that relation (a) is impossible.

We note that if c^1 is any 1-cell of Γ^1, then every point of the transform, K^1, of c^1, is at a distance less than $\frac{1}{2}\epsilon$ from either end-cell of c^1. That is, both of the chains c^1 and K^1 lie within a spherical neighborhood T of radius ϵ which has its center at one end-cell of c^1 and accordingly contains no point of

J. By introducing, if necessary, two new 1-chains, each bounded by an end-cell of c^1 and its transform, a closed 1-chain L^1 is obtained in T which contains c^1 and K^1 and such that

(b) $$L^1 \sim 0 \qquad (\text{mod } 2, T, E_3 - J).$$

By adding homology (a) and all homologies of type (b), we have

(c) $$\Gamma_1{}^1 + \sum L^1 \sim 0 \qquad (\text{mod } 2, E_3 - J).$$

However, we have that, adding modulo 2,

(d) $$\Gamma_1{}^1 + \sum L^1 = \Gamma^1$$

(it can be assumed that only one chain is introduced between each 0-cell of Γ^1 and its transform). Combining relations (c) and (d) we have

$$\Gamma^1 \sim 0 \qquad (\text{mod } 2, E_3 - J).$$

This contradicts the fact that Γ^1 links J. Hence Γ^1 must have points in both S_1 and S_2.

We shall now show that the supposition that $K-J$ is connected is inconsistent with the fact that Γ^1 links J, and has points in both S_1 and S_2. It will first be necessary to make a transformation of Γ^1.

Let ϵ_1 be a positive number less than $\frac{1}{4}\epsilon$. Since K is uniformly connected im kleinen (being a bounded, connected im kleinen continuum), there is a positive number ϵ_2, such that if P and Q are points of K at a distance apart less than ϵ_2, then P and Q are joined by an arc of K of diameter less than ϵ_1. Let δ_{ϵ_2} be a positive number such that if P and Q are 0-cells of S_i ($i=1$ or 2) at a distance apart less than δ_{ϵ_2}, then P and Q bound a 1-chain in S_i of diameter less than $\frac{1}{4}\epsilon_2$. We may assume that the 1-cells of Γ^1 are all of diameter less than $\frac{1}{4}\delta_{\epsilon_2}$.

The transformation that we now effect on Γ^1 is very similar to the one described above, except that the new constants just defined are involved. Supposing that we start with a cell c^1 of Γ^1 whose end-cells are both in S_1, we replace it by a chain-cell K^1 with the same boundary and diameter less than $\frac{1}{4}\epsilon_2$. If the next cell b^1 has an end-cell on K, we transform it into a chain-cell in S_1. Of course we ultimately come to a cell c^1 one of whose end-cells (the one last affected by a transformation), a^0, is in S_1+K, and the other, b^0, is in S_2. (Indeed, if Γ^1 had none of its bounding 0-cells in S_2, the whole chain Γ^1 could be transformed into a chain in S_1, just as in the above proof, and a contradiction obtained as before. Similarly, if we encounter a case of one bounding 0-cell of Γ^1 in S_2 (S_1) and the neighboring 0-cells in S_1 (S_2), the 1-cells which they bound may be transformed into chain-cells lying in

$S_1\,(S_2)$.) It is unnecessary to use condition (3) here, since any 1-chain K^1 of diameter less than $\frac{1}{4}\epsilon_2$ and bounded by $\bar{a}^0$ and b^0 will do for the transform of c^1 ($\bar{a}^0$ being the transform of a^0). The next 0-cell, c^0, is in S_2 (else we would have proceeded as indicated in the above parenthesis) and the transformation to be effected is obvious. The subsequent transformations should also be obvious, since we now proceed from S_2 in just such a manner as we proceeded from S_1, viz., by not actually "crossing" into S_1 until two successive bounding 0-cells are encountered in S_1.

The outcome of these transformations on the cells of Γ^1 is to obtain a new 1-chain, which we shall still denote by Γ^1 and which still links J (this can be shown as above), but which has the following properties: (1) its intersections with K occur on certain chain-cells, $K_1^1, K_2^1, \cdots, K_n^1$ in the order named; we shall henceforth call these the *crossing-cells* of Γ^1; (2) if K_i^1 is any crossing-cell, the diameter of K_i^1 is less than $\frac{1}{4}\epsilon_2$; (3) the boundary cells of K_i^1 lie in S_1 and S_2, respectively; (4) if K_i^1 and K_{i+1}^1 are bounded by the cells a^0, b^0, c^0 and d^0 (these occurring in the order named), then the cells a^0 and d^0 are both in $S_1(S_2)$; in other words, if one crossing-cell leads from S_1 to S_2, the next leads from S_2 to S_1. The last property is very important, and shows that there is an even number of crossing-cells.

We shall now proceed to replace Γ^1 by a 1-chain which still links J but has only *two* crossing-cells.

Starting with K_1^1, suppose that the bounding 0-cells of K_1^1 and K_2^1 occur in the order a^0, b^0, c^0, d^0, where a^0 is in S_1. In S_1, let A_1^1 be a new 1-chain bounded by a^0 and d^0, and in S_2 let A_2^1 be the 1-chain of Γ^1 which lies in S_2 and is bounded by b^0 and c^0. Denote the portion of Γ^1 from d^0 to a^0 by B^1. Now we cannot have both of the following relations:

$$\text{(e)} \qquad A_1^1 + K_1^1 + A_2^1 + K_2^1 \sim 0,$$

$$\text{(f)} \qquad A_1^1 + B^1 \sim 0 \qquad (\text{mod } 2,\ E_3 - J),$$

since the sum of these homologies would imply the homology

$$\Gamma^1 \sim 0 \qquad (\text{mod } 2,\ E_3 - J).$$

If (e) fails to hold, we have secured the type of 1-chain linking J that we set out to obtain, viz., a chain with only two crossing-cells. If (e) holds, we can proceed with the 1-chain

$$\Gamma^1 + K_1^1 + A_2^1 + K_2^1 + A_1^1 = A_1^1 + B^1 \qquad (\text{mod } 2)$$

which has two less crossing-cells than Γ^1, viz., $K_3^1, \cdots, K_n^1$, by next considering the crossing-cells K_3^1 and K_4^1.

In any case, we eventually arrive at a closed 1-chain which links J, and has only two of the original crossing-cells of Γ^1. We shall continue to call this chain Γ^1, and we shall suppose its crossing-cells are K_1^1 and K_2^1, their boundary cells being denoted, as before, by a^0, b^0, c^0, d^0. Denote the portion of Γ^1 lying in S_1 and bounded by a^0+d^0 by L_1^1 and the portion in S_2 bounded by b^0+c^0 by L_2^1.

Because of the way in which the above transformation was carried out, at least one of the boundary cells of K_i^1 $(i=1, 2)$ is identical with a cell of the original chain Γ^1, so that a spherical neighborhood T_i of diameter ϵ_2 about K_i^1 certainly encloses no points of J. We may assume that K_1^1 and K_2^1 are homeomorphic with simple arcs, and on K_i^1 $(i=1, 2)$ let x_i and y_i be points of K such that the portions a^0x_1 (d^0x_2) and b^0y_1 (c^0y_2) of K_1^1 (K_2^1) contain no points of K other than x_1 and y_1 (x_2 and y_2).

Since x_1 and x_2 lie in $K-J$, and $K-J$ is connected, there is an arc x_1x_2 in $K-J$.* Denote the distance between x_1x_2 and J by ϵ_3, let ϵ_4 denote a positive number less than both ϵ_1 and ϵ_3, and let δ_{ϵ_4} be a positive number such that if two 0-cells of S_i are at a distance apart less than δ_{ϵ_4}, they bound a 1-chain in S_i of diameter less than $\frac{1}{4}\epsilon_4$ (condition (3)).

Let $F_1, F_2, \cdots, F_m$ be points on the arc x_1x_2, where $F_1=x_1$ and $F_m=x_2$, occurring in the order named, and so selected that

$$\delta(F_iF_{i+1}) < \tfrac{1}{4}\delta_{\epsilon_4} \qquad (i = 1, 2, \cdots, m-1),$$

where F_iF_{i+1} denotes the portion of x_1x_2 between F_i and F_{i+1}. Let $P_1, P_2, \cdots, P_m$ be a set of 0-cells in S_1 such that P_1 is on the portion a^0x_1 of K_1^1 and P_m on the portion d^0x_2 of K_2^1, and the following relations hold:

$$\rho(P_i, F_i) < \tfrac{1}{4}\epsilon_4,$$
$$\rho(P_i, P_{i+1}) < \delta_{\epsilon_4}.$$

According to the definition of δ_{ϵ_4}, there exists for each i an open 1-chain H_i^1 in S_1 bounded by P_i and P_{i+1}, and such that $\delta(H_i^1)<\frac{1}{4}\epsilon_4$. We define $C_1^1=\sum_{i=1}^{m-1} H_i^1$, and we also note that every point of H_i^1 lies within a distance $\frac{1}{2}\epsilon_4$ of F_i.

Let T_i^1 $(i=1, 2)$ be a spherical neighborhood concentric with T_i and of diameter $2\epsilon_1$. Clearly T_i^1 contains no point of J. According to the definition of ϵ_2, and because of the fact that the distance from x_i to y_i is less than ϵ_2 (the diameter of K_i^1 being less than ϵ_2), there is an arc t_i in K which lies wholly in T_i and has end points x_i and y_i. We can approximate the arc t_1 by a $\frac{1}{4}\delta_{\epsilon_4}$-chain† of 0-cells in S_2, with first point, z_1, on the portion y_1b^0 of K_1^1, and

* R. L. Moore, Mathematische Zeitschrift, loc. cit.

† The word "chain" is here used in the ordinary point-set theoretic sense; i.e., if ϵ is a positive number, and P and Q are points, an ϵ-chain from P to Q is a set of points $P_1, P_2, \cdots, P_n$, where $P_1=P$ and $P_n=Q$, and $\rho(P_i, P_{i+1})<\epsilon$ $(i=1, 2, \cdots, n-1)$.

last point, z_h, and such that not only is every point of the chain at a distance less than $\frac{1}{4}\epsilon_4$ from some point of t_1, but in particular $\rho(z_h, F_1)<\frac{1}{4}\epsilon_4$. Then, in a manner similar to that used in obtaining C_1^1, we obtain a 1-chain B_1^1 bounded by z_1+z_h, lying wholly in S_2, and such that every point of B_1^1 is within a distance $\frac{1}{2}\epsilon_4$ from some point of t_1. It is clear that a spherical neighborhood T_1^2 concentric with T_1 and of diameter $3\epsilon_1$, will contain B_1^1, and enclose no points of J.

We can obtain similarly a 1-chain B_2^1 with reference to t_2, so that the boundary of B_2^1 is a point w_1 on the portion y_2c^0 of K_2^1 and a point w_k in S_2 whose distance from F_m is $<\frac{1}{4}\epsilon_4$. Let T_2^2 be a spherical neighborhood analogous to T_1^2.

Let $Q_1, Q_2, \cdots, Q_m$ be points in S_2 chosen with reference to the points $\{F_i\}$ just as the points $\{P_i\}$ were chosen in S_1, and so that $Q_1=z_h$ and $Q_m=w_k$. On these points can be built up a 1-chain B_3^1 of S_2, bounded by Q_1 and Q_m, and approximating the arc x_1x_2 in a manner similar to that in which C_1^1 approximates x_1x_2.

We now define the following 1-chains:

$$\begin{aligned} C_2^1 &= \sum_{i=1}^{3} B_i^1, \\ \Gamma_1^1 &= L_1^1 + a^0P_1 + C_1^1 + P_md^0, \\ \Gamma_2^1 &= C_1^1 + P_1z_1 + C_2^1 + w_1P_m, \\ \Gamma_3^1 &= C_2^1 + z_1b^0 + L_2^1 + c^0w_1, \end{aligned}$$

where a^0P_1, P_md^0, etc., are portions of K_1^1 and K_2^1. We note that

$$\text{(g)} \qquad \Gamma^1 = \sum_{i=1}^{3}\Gamma_i^1 \qquad (\text{mod } 2).$$

The following homologies follow from condition (4) of the theorem:

$$\text{(h)} \qquad \Gamma_1^1 \sim 0,$$

$$\text{(i)} \qquad \Gamma_3^1 \sim 0 \qquad (\text{mod } 2, E_3 - J).$$

Consequently, since Γ^1 links J, it follows from relations (g), (h) and (i) that Γ_2^1 links J.

If we join P_1 and z_h by a 1-chain B_4^1 every point of which is at a distance less than $\frac{1}{2}\epsilon_4$ from x_1 $(=F_1)$ and P_m and w_k similarly by a 1-chain B_5^1, it is easy to see that since the following relations hold:

$$B_1^1 + B_4^1 + P_1z_1 \sim 0 \qquad (T_1^2, E_3 - J),$$

$$B_2^1 + B_5^1 + P_mw_1 \sim 0 \qquad (T_2^2, E_3 - J),$$

the closed chain

$$\Gamma = C_1^1 + \sum_{i=1}^{5} B_i^1$$

links J.

However, this is impossible for the following reasons. The points P_i and Q_i ($i=2, 3, \cdots, m-1$) bound a 1-chain N_i^1, every point of which is at a distance less than $\frac{1}{4}\epsilon_4$ from F_i. For convenience of notation, let $N_1^1 = B_4^1$ and $N_m^1 = B_5^1$. Let U_i ($i=1, 2, \cdots, m$) be a spherical neighborhood of F_i of radius ϵ_4. Then every point of the closed chain

$$E_i^1 = H_i^1 + N_i^1 + \overline{H}_i^1 + N_{i+1}^1,$$

where $\overline{H}_i^1$ is the portion of B_3^1 from Q_i to Q_{i+1}, lies in U_i. Hence

$$\text{(j)} \qquad E_i^1 \sim 0 \qquad (\text{mod } 2, E_3 - J).$$

But

$$\Gamma = \sum_{i=1}^{m} E_i^1 \qquad (\text{mod } 2),$$

and by adding all homologies (j), we get

$$\Gamma \sim 0 \qquad (\text{mod } 2, E_3 - J).$$

Thus, on the assumption that $K-J$ is connected, we are able to find a closed chain Γ, homologous to the chain Γ^1 which links J, but which does not itself link J. As this is impossible, $K-J$ cannot be connected.

VI. *If J is a simple closed curve of K, then $K-J$ contains at most two components.* For suppose $K-J$ contains at least 3 components, M_1, M_2, and M_3. Since K is a continuous curve, each component of $K-J$ is also a quasi-component of $K-J$,* and it is therefore easy to see, with the result of section IV, that each component of $K-J$ has all of J as its boundary.

There exists on J a set of points occurring in the following order, $P_1A_1Q_1B_1P_2B_2Q_2A_2P_1$, and such that there is an arc $A_ix_iB_i$ ($i=1, 2$) lying, except for A_i and B_i, wholly in M_i. Define a simple closed curve J_1 as follows:

$$J_1 = \sum_{i=1}^{2} A_ix_iB_i + A_1P_1A_2 + B_1P_2B_2,$$

where $A_1P_1A_2$ and $B_1P_2B_2$ are arcs of J. Then the set $K-J_1$ is connected, in contradiction to the result of section V. For if we denote the set

* Cf. R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11 (1928), pp. 127–131.

$K-(M_1+M_2+J)$ by R, it is clear that the set $R_1=R+\sum_{i=1}^2<A_iQ_iB_i>$ is connected. Also, the set $M_1-<A_1x_1B_1>$ contains only components that have boundary points on either $<A_1Q_1B_1>$ or $<A_2Q_2B_2>$. For suppose $M_1-<A_1x_1B_1>$ contains a component C with boundary points only on the set $t=A_2P_1A_1x_1B_1P_2B_2$. Then the set

$$K-t=C+\left[(M_1-<A_1x_1B_1>-C)+M_2+R+\sum_{i=1}^{2}<A_iQ_iB_i>\right]$$

is not connected, contrary to the result of IV. In a like manner it is shown that all components of $M_2-A_2x_2B_2$ have limit points on $\sum_{i=1}^2<A_iQ_iB_i>$, and it is clear that the set

$$K-J_1=R_1+\sum_{i=1}^{2}(M_i-A_ix_iB_i)$$

is connected.

VII. *Denoting the two components of $K-J$ by M_1 and M_2, respectively, the sets M_1 and M_2 are uniformly connected im kleinen.* For suppose M_1 is not uniformly connected im kleinen. Then there is a point P on J, a spherical neighborhood R of P, and a sequence of pairs of points of M_1, $\{P_n, Q_n\}$, such that $\lim_{n\to\infty}P_n=P$ and $\lim_{n\to\infty}Q_n=P$, and for no n are P_n and Q_n joined in R by any connected subset of M_1. Denote the set $M_i\cdot R$ by $\overline{M}_i$ ($i=1, 2$), and the frontier of R by F.

Denote by C the component of $K\cdot R$ determined by P. Let R_1 be a spherical neighborhood of P of smaller radius than R, which encloses no points of J that are not on the arc, t, component of $J\cdot R$ determined by P. We shall denote $J\cdot R$ by t_1.

Let R_2 be another spherical neighborhood of P such that any point of K within R_2 is joined to P by an arc of K that lies in R_1 and such that R_2 encloses no points of $K-C$. Let P_k, Q_k be a pair of points of the sequence $\{P_n, Q_n\}$ lying in R_2, and let the component of $\overline{M}_1$ determined by P_k be denoted by H_1; let $\overline{M}_1-H_1=H_2$. Then H_1 and H_2 are mutually separated.

In R_2, let x_1 and x_2 be points of S_1 and S_2, respectively, that are 0-cells of the subdivision of E_3.

(a) The set C separates R between x_1 and x_2. For x_1 and x_2 bound a chain Γ_1^1 in $E_3-[F+(K-C)]$, and if they bound a chain Γ_2^1 in $E_3-(F+C)$, then, since the closed chain $\Gamma_1^1+\Gamma_2^1$ is homologous to zero in R, it cannot link the product $[F+(K-C)]\cdot[F+C]$, which is a subset of F, and accordingly x_1+x_2 is homologous to zero in E_3-K (Alexander's Corollary W_i, loc. cit.). This is a contradiction of condition (1) of the theorem.

(b) Under the assumption that M_1 is not uniformly connected im kleinen in the neighborhood of P, C does not separate x_1 and x_2 in R. Let y_1 and y_2 be the first points of K on a straight line from x_1 to x_2, in the orders x_1 to x_2 and x_2 to x_1, respectively. There exist arcs y_1P_k, y_1Q_k, y_2P_k and y_2Q_k of K in R_1. By the use of the usual approximation based on condition (3) of the theorem, we obtain a closed 1-chain, Γ^1, consisting of two open chains Γ_1^1 and Γ_2^1 bounded by x_1 and x_2, such that Γ^1 lies in R_1, and cuts K only in certain arbitrarily small neighborhoods of P_k and P_j, respectively. The following congruences hold:

$$\Gamma_1^1 \equiv x_1 + x_2 \qquad [\text{mod } 2, E_3 - (F + H_1')],$$

$$\Gamma_2^1 \equiv x_1 + x_2 \qquad [\text{mod } 2, E_3 - (F + H_2' + \overline{M}_2')]$$

(where H_1' denotes H_1 together with all its limit points, etc.). We may assume that the product of H_1' and H_2' is a subset of t_1', so that if we can show that Γ^1 bounds in $E_3-(F+H_1')\cdot(F+H_2'+\overline{M}_2')$ or, since the latter set contains $E_3-(F+t_1)$, if we can show that Γ^1 bounds in $E_3-(F+t_1)$, we shall have that x_1 and x_2 are not separated in R by C (Alexander's Corollary W_i, loc. cit.).

Since M_1 is connected, it can be shown, by the methods employed in section V, that Γ^1 does not link J. Consequently, there exists a chain Γ_1^2 such that

$$\Gamma_1^2 \equiv \Gamma^1 \qquad (\text{mod } 2, E_3 - J, E_3 - t').$$

But since Γ^1 lies in R_1, and the latter contains no points of $F+J-t$, we have

$$\Gamma_2^2 \equiv \Gamma^1 \qquad [\text{mod } 2, E_3 - (F + J - t)].$$

Now since the product of $F+J-t$ and t' is just two points, and neither Γ_1^2 nor Γ_2^2 meets the arc $J-t$ joining these points, we have that

$$\Gamma_1^2 + \Gamma_2^2 \sim 0 \quad [\text{mod } 2, E_3 - t'\cdot(F + J - t)].$$

Hence, by Alexander's Corollary W_i (loc. cit.),

$$\Gamma^1 \sim 0 \qquad [\text{mod } 2, E_3 - (F + t_1)].$$

As this is the relation we wished to prove, in order to show that the 0-chain x_1+x_2 bounds in $R-C$, we have shown that x_1 and x_2 are not separated by C in R.

As (a) and (b) are in contradiction, the assumption that M_1 is not uniformly connected im kleinen cannot hold. A like statement of course holds for M_2, and the theorem is proved.

The following theorem follows immediately from Theorem 3, from which it is only slightly different:

THEOREM 4 (Converse of the Jordan-Brouwer Separation Theorem in E_3). *Let K be a closed and bounded set in E_3, such that* (1) *the Betti numbers* (mod 2) $R^0(E_3-K)$ *and* $R^1(E_3-K)$ *are respectively equal to* 2 *and* 1, (2) *if D is a component of E_3-K, the* 0*-chains in D are uniformly homologous to zero in D, and every point of K is a limit point of D. Then K is a simple closed surface.*

Since there are two components, S_1 and S_2, in E_3-K, condition (1) of Theorem 3 is satisfied. Since every point of K is a limit point of each of these components, condition (2) of Theorem 3 is satisfied; for, it is to be noted, condition (2) of Theorem 3 allows P' to lie in that complementary domain which contains Q. Conditions (3) and (4) of Theorem 3 are restated in Theorem 4. Thus Theorem 4 follows at once from Theorem 3.

4. **Appendix.** The independence of the conditions of Theorem 3 is established as follows:

Condition (1): In E_3, using rectangular coördinates, let K be the set of points $(x, y, 0)$ such that $x^2+y^2\leqq 1$. Let S_1 be the set of all points for which $z>0$; let $S_2=E_3-(K+S_1)$.

Condition (2): Using spherical coördinates, let K be the set of all points (ρ, ϕ, θ) for which $1\leqq\rho\leqq 2$, let S_1 be the set of points for which $\rho<1$, and S_2 the set for which $\rho>2$.

Condition (3): Using spherical coördinates, let K' denote the set of points $\rho=1$. On K', let t denote the arc consisting of points $(1, 0, \theta)$ such that $0\leqq\theta\leqq\pi$. On t, if P denotes any point $(1, 0, \theta)$, let P' denote the point $(1, 0, \pi-\theta)$. The set K is obtained by continuously deforming K' so that each point P coincides with P', but points not on t remain distinct as before. The complement of K is two domains, neither of which satisfies condition (3) of Theorem 3.

Condition (4): The torus.

The equivalence, in the case of open sets in E_n, of the conditions *uniformly homologous to zero* and *uniformly connected im kleinen*, is established in the following theorem:

THEOREM 5. *In order that an open set G in E_n should be uniformly connected im kleinen, it is necessary and sufficient that the* 0*-chains in G should be uniformly homologous to zero in G.*

The condition is necessary. Let ϵ be any positive number. Since G is uniformly connected im kleinen, there exists a positive number δ such that if P and Q are 0-cells of G whose distance apart is less than δ, there is an arc PQ of G whose diameter is less than $\frac{1}{4}\epsilon$.

Let $E_n - G = F$, and denote the distance between F and the arc PQ by η_1. Let η be a positive number smaller than either of the numbers η_1, $\frac{1}{2}\epsilon$. Let $P_1(=P), P_2, \cdots, P_n(=Q)$ be points of PQ such that $\rho(P_i, P_{i+1}) < \frac{1}{4}\eta (i=1, 2, \cdots, n-1)$. The subdivision of E_n may be extended so that there exist 0-cells $F_1(=P_1), F_2, \cdots, F_n(=P_n)$ such that $\rho(F_i, F_{i+1}) < \frac{1}{4}\eta$ and $\rho(F_i, P_i) < \frac{1}{4}\eta$. Let T_i be a spherical neighborhood of P_i of radius $\frac{1}{2}\eta$. Then clearly F_{i+1} lies in T_i, and consequently $F_i + F_{i+1}$ bounds a 1-chain H_i in T_i. Since no point of F lies in T_i, $H_i \cdot F = 0$. Consequently the chain $K^1 = \sum_{i=1}^{n-1} H_i$ lies in G.

The diameter of K^1 is less than ϵ. For let x_1 and x_2 be points of K^1. Since x_1 lies in some sphere T_i, and hence $\rho(x_1, P_i) < \frac{1}{2}\eta < \frac{1}{4}\epsilon$, and since there exists, similarly, a P_j such that $\rho(x_2, P_j) < \frac{1}{4}\epsilon$, it follows at once from the fact that the diameter of PQ is $< \frac{1}{4}\epsilon$, that $\rho(x_1, x_2) < \epsilon$.

The condition is sufficient. If ϵ is a positive number, there is a positive number δ such that if x_1^0 and x_2^0 are 0-cells of G such that $\rho(x_1^0, x_2^0) < \delta$, then $x_1^0 + x_2^0$ bounds a chain K^1 of G of diameter $< \frac{1}{4}\epsilon$. Let P_1 and P_2 be any two points of G such that $\rho(P_1, P_2) < \frac{1}{4}\delta$. The subdivision of E_n may be extended so that there exist 0-cells y_1^0 and y_2^0 such that $\rho(x_i^0, P_i) < \frac{1}{4}\delta$ $(i=1, 2)$, and such that there are arcs y_iP_i of diameter $< \frac{1}{4}\epsilon$ in G. Since $\rho(y_1^0, y_2^0) < \delta$, $y_1^0 + y_2^0$ bounds a chain K^1 of G of diameter $< \frac{1}{4}\epsilon$. The set $K^1 + \sum_{i=1}^{2} y_iP_i$ contains an arc from P_1 to P_2 of diameter $< \epsilon$.

As a result of Therem 5 we can restate Theorem 4 as follows:

THEOREM 4′(Converse of the Jordan-Brouwer Separation Theorem in E_3). *In E_3, the common boundary, K, of two uniformly connected im kleinen domains, one of which is bounded, is a simple closed surface, provided that the Betti number* (mod 2) $R^1(E_3 - K) = 1$.

The problem in n dimensions, $n > 3$. Regarding a converse of the Jordan-Brouwer Separation Theorem in E_n, where $n > 3$, the author will not hazard any guesses here, but merely indicate the likelihood that by an extension of the conditions of Theorem 4, such a converse may be obtained. Thus, condition (1) of Theorem 4 may be replaced, as a result of the Alexander duality theorem, by the condition that the Betti numbers $R^0(E_3 - K)$ and $R^i(E_3 - K)$ $(i = 1, 2, \cdots, n-2)$ shall be respectively equal to 2 and 1, and condition (2) may be replaced by the statement that the closed i-chains $(i = 0, 1, \cdots, n-2)$ of D are uniformly homologous to zero in D, retaining, of course, the statement that every point of K is a limit point of D. The validity of such a condition (2) is established by the following theorems, with which we close the present paper. (The extension of the definition of "uniformly homologous to zero" to i-chains where $i > 0$ should be obvious.)

THEOREM 6. *Let M^{n-1} be an $(n-1)$-sphere immersed in E_n, and let D be one of the domains of $E_n - M^{n-1}$. Then the closed i-chains $(n>i>0)$ of D are uniformly homologous to zero in D.*

Suppose the conclusion of the theorem untrue. Then there is a point P of M^{n-1} and a contracting sequence of i-chains of D, $K_1^i, K_2^i, K_3^i, \cdots$, such that P is the sequential limit point of any set of points $P_1, P_2, P_3, \cdots$, where P_n, for every n, is a point of K_n^i, and such that there is a positive number ϵ for which K_n^i, for every n, fails to bound any $(i+1)$-chain of D of diameter less than ϵ.

Let $S(P, \epsilon/2)$ be a spherical neighborhood of P of diameter $\epsilon/2$. Then there is an $(n-1)$-cell, C^{n-1}, of M^{n-1}, which contains P as an interior point and lies wholly in $S(P, \epsilon/2)$. Let R denote a spherical neighborhood of P which encloses only points of M^{n-1} that lie interior to C^{n-1}. Denote the boundaries of $S(P, \epsilon/2)$ and C^{n-1} by $F(P, \epsilon/2)$ and C^{n-2}, respectively. The set C^{n-2} is of course an $(n-2)$-sphere immersed in E_n.

Let us first consider the case where $i<n-1$.

There is a number j, such that K_j^i lies wholly in R. Then, denoting by B^{n-1} the set of all points of M^{n-1} that are not interior points of C^{n-1}, we have the following congruences:

$$L_1^{i+1} \equiv K_j^i \qquad (\text{mod } 2, E_n - C^{n-1}),$$

$$L_2^{i+1} \equiv K_j^i \qquad \{\text{mod } 2, R, E_n - [B^{n-1} + F(P, \epsilon/2)]\}.$$

Since $i>0$, $i+1$ is greater than 1, and since by Alexander's Theorem X^i an $(n-2)$-sphere immersed in E_n can be linked only by a 1-chain, the following homology holds:

$$L_1^{i+1} + L_2^{i+1} \sim 0 \qquad (\text{mod } 2, E_n - C^{n-2}).$$

As C^{n-2} is the common part of C^{n-1} and $B^{n-1}+F(P, \epsilon/2)$, it follows from Alexander's Corollary W^i that

$$K_j^i \sim 0 \qquad [\text{mod } 2, D \cdot S(P, \epsilon/2)].$$

As this contradicts our assumption, the theorem is proved for $i<n-1$.

If $i=n-1$, the proof is trivial. Each of the chains K_j^i may be assumed irreducible (i.e., $R^i(K_j^i)=2$), and thus, by Alexander's Theorem Y, separates E_n into just two domains, one of which, D_1, must lie in R. The domain D_1 is an open n-chain bounded by K_j^i and containing no point of M^{n-1}.

A similar proof, together with the result of Theorem 1a, yields the following general result:

THEOREM 7. *Let M^{n-1} be any $(n-1)$-manifold immersed in E_n, D a domain complementary to M^{n-1}, and P a point of M^{n-1} which can be considered as* *interior to an* $(n-1)$*-cell of definition of* M^{n-1}. *Then if* U *is a neighborhood of* P *bounded by an* $(n-1)$*-sphere and enclosing only points of* M^{n-1} *that are interior to* C^{n-1}, *there exists, in* U, *a neighborhood* V *of* P *such that any i-chain* $(i=0, 1, \cdots, n-1)$ *that lies in* $D\cdot V$ *bounds an* $(i+1)$*-chain of* $D\cdot U$. *If* M^{n-1} *is of the type such that every one of its points can be considered as lying interior to an* $(n-1)$*-cell of definition of the manifold, then the i-chains* $(i=0, 1, \cdots, n-1)$ *of* D *are uniformly homologous to zero in* D.

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